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Résumé

We show that the supremum for c real of the Hausdorff dimension of the Julia set of the polynomial $z \mapsto z^d + c$ (d is an even natural number) is greater than $2d/(d+1)$.

1 Introduction and statement of the result

The Julia set of a non-linear polynomial $P : \mathbf{C} \to \mathbf{C}$ is the set of points having no neighborhood on which the family of iterates (P^n) is normal. It is a compact non-empty set which is, except for very special polynomials P, a fractal set. For $c \in \mathbf{C}$ we denote by f_c the polynomial

$$f_c(z) = z^d + c,$$

where $d \geq 2$ is an even integer number, which we fix. Denote by J_c the Julia set of f_c.

In the quadratic case $d = 2$, the values of c for which the Hausdorff dimension (HD) of J_c is big has attracted a lot of attention. It has been known since the pioneering work of Douady-Hubbard [3] that the Hausdorff dimension of the Julia set is less than 2 for every hyperbolic polynomial. Thus $HD(J_c) < 2$ outside ∂M and the (hypothetic) non-hyperbolic components of (the interior) M. We recall that M stands for the Mandelbrot set, that is the compact subset of $c \in \mathbf{C}$ such that J_c is connected. Shishikura [9] was the first to find quadratic Julia sets with Hausdorff dimension 2. He indeed proved that this property holds on a dense $\mathcal{G}_\delta$ subset of ∂M or even on a dense $\mathcal{G}_\delta$ subset on the boundary of every hyperbolic component of M.

More recently, Buff and Chéritat [12] have found quadratic Julia sets with positive Lebesgue measure (see also http ://annals.math.princeton.edu/articles/3682).

Both Shishikura's and Buff-Chéritat's results are based on the phenomenon of parabolic implosion which has been discovered and studied by Douady-Hubbard [3]. It should be pointed out that Buff-Chéritat's result is very involved and that we will make no use of it. There is no doubt that if they exist, values of $c \in \mathbf{R}$ such that the Julia set J_c has positive measure must be as hard to find as Buff-Cheritat's ones. The aim of present note is much more modest. Its starting point is the second author re-readind of Shishikura's result [10] : it states that if one implodes a polynomial with a parabolic cycle having q petals, then the dimension of its Julia set automatically gets bigger than $2q/(q+1)$. Shishikura's result follows from this by a Baire argument.

Very little is known about Hausdorff dimension of J_c for real c. In particular, it is not known if, for a given degree d,

$$\sup\{HD(J_c), c \in \mathbf{R}\} = \mathbf{2}.$$

Possible candidates for high dimension are of course (just look at them !) infinitely renormalizable polynomials but the analysis seems to be very delicate and at least no result concerning dimension 2 has been proven so far (for results in the opposite direction, see [1] though). It is for example unknown if the Julia set of the Feigenbaum polynomial has dimension 2 or not (see [4]- [5] for the Julia set of the Feigenbaum universal map though). The only known result about this set is a very general result of Zdunik [14] : it has dimension bigger than 1.

If one tries to use the same ideas as in [9] for real polynomials f_c, one immediately faces the problem that if f_c, for c real, has a parabolic cycle that may be imploded along the real axis then the number of petals is 1 and this does not imply more than Zdunik's general result. The only trick of this paper is to make use of a virtual doubling of petals when the critical point is mapped to a parabolic point (by Lavaurs map). It was inspired by Douady et al paper [13] and implies the following theorem, which is the main result of this work.

Theorem 1 *Let $f_c(z) = z^d + c$, d even. Let N be the set of parameters $c \in \mathbf{R}$, such that f_c has a parabolic cycle of period at least 2 and multiplier 1. Then there exists an open set Y of $\mathbf{R}$ whose closure contains N such that J_c is connected and*

$$HD(J_c) > \frac{2d}{d+1},$$

for every $c \in Y$.

Comment 1 *In fact, we prove a stronger statement : hyperbolic dimension [9] of J_c is bigger than $2d/(d+1)$.*

Comment 2 *By [11] ([6], [8] for $d = 2$), the set of real c such that f_c is hyperbolic is dense in $\mathbf{R}$. In particular, hyperbolic parameters c are dense in Y.*

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2 Proof of the theorem

We fix an even integer $d \geq 2$ and consider the family $f_c(x) = x^d + c$, for real c. Then the Julia set J_c is connected if and only if $c \in [a, b]$, where $a < 0$ is such that $f_a^2(0)$ is a fixed point, and $b > 0$ is such that f_b has a fixed point with multiplier 1. It is sufficient to prove that, given $c_0 \in (a, b)$ such that f_{c_0} has a neutral cycle of period $k > 1$ with multiplier 1, that is, with one petal, there is an open set Z accumulating at c_0 for which $HD(J_c) > 2d/(d+1)$ for $c \in \mathbf{Z}$.

We begin with three pictures. The first two ones illustrate how to choose the parameter (for $d = 2$) and how the corresponding Julia set looks like. The third one has been kindly drawn for us by the referee and shows the case $d = 4$.

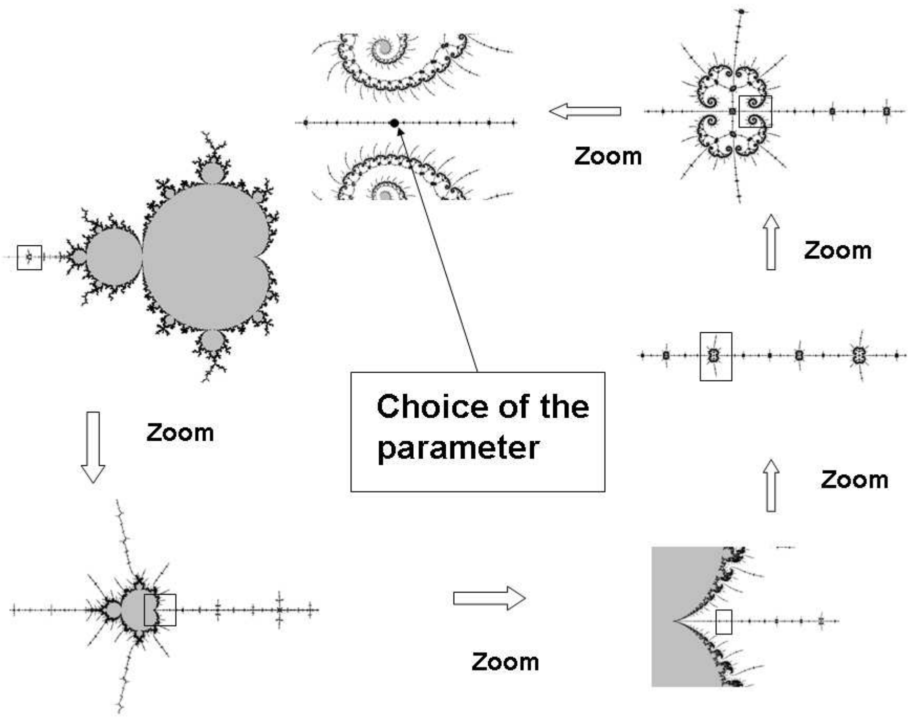

FIGURE 1 – Choice of the parameter

Since $c_0 \in (a, b)$, the set J_c is connected for c in a small neighborhood U of c_0. It is known, that f_c has an attracting periodic orbit of period k for $c \in U$ on the left side of c_0. Since $k > 1$, the corresponding filled-in Julia set K_{c_0} is such that

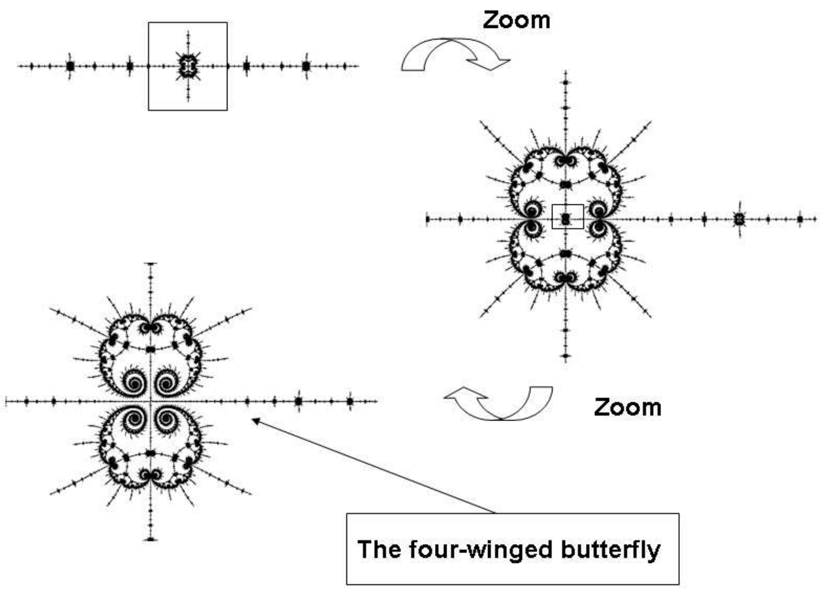

FIGURE 2 – The corresponding Julia set

its interior has a component Δ containing 0, and infinitely many preimages of Δ. The boundary $\partial\Delta$ contains a parabolic point α of period k. In particular, there is a sequence of preimages of Δ, which intersect the real line and accumulate at α.

For definicity, one can assume that $\alpha > 0$. Then $F = f_{c_0}^k$ has the following local form :

$$F(z) = z + a(z-\alpha)^2 + b(z-\alpha)^3 + ..., \tag{1}$$

where $a > 0$. This implies that a parabolic implosion phenomenon occurs as $\epsilon \to 0$, $\epsilon > 0$, for the maps $f_{c_0+\epsilon}$.
At this point we digress somewhat and describe briefly the theory of parabolic implosion.
Let $\delta > 0$ be very small and $D_\pm$ be the disks centered at $\alpha \pm \delta$ with radius δ. The map F sends D_- into itself while $D_+ \subset F(D_+)$. This defines, after identification of z with $F(z)$ at the boundary, two cylinders $U_- = D_- \backslash F(D_-)$ and $U_+ = F(D_+) \backslash D_+$. The fact that $U_\pm$ are actual cylinders is best seen in the approximate Fatou coordinate $I : z \mapsto -1/(a(z-\alpha))$ which sends α to ∞ and

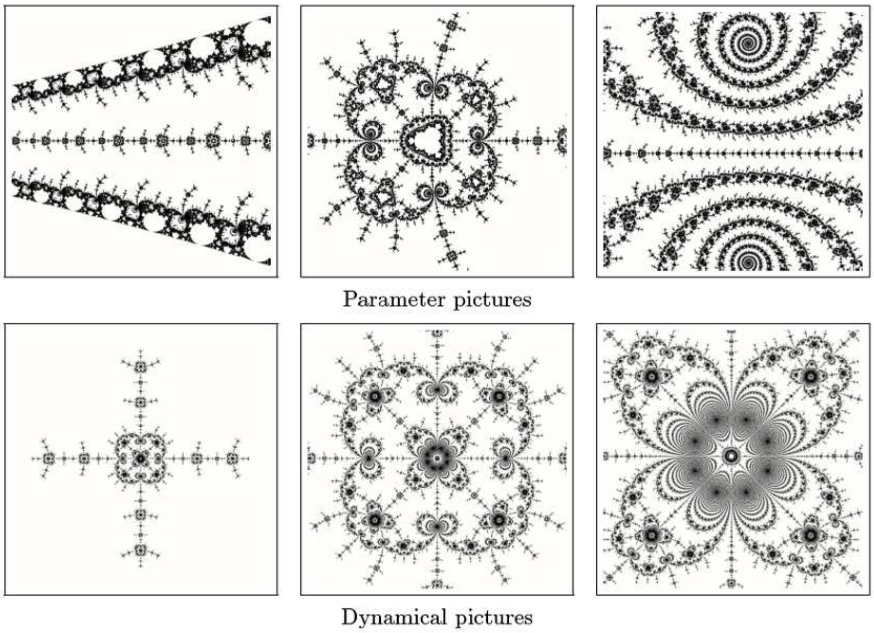

FIGURE 3 – The case d=4

conjugates F to a map which is asymptotically the translation by 1 at ∞ :

$$F_\infty(w) = I \circ F \circ I^{-1}(w) = w + 1 + \frac{A}{w} + O(\frac{1}{|w|^2}), \tag{2}$$

where $A = 1 - b/a^2$. The real number A is a conformal invariant. In the case of real polynomial F which has a parabolic fixed point α with multiplier 1 and with a single critical point in its immediate basin Δ, it is known [9] that $A > 0$.

By Riemann mapping theorem these two cylinders may be uniformized by "straight" cylinders. In other words there exists $\varphi_\pm$ mapping the cylinders $U_\pm$ to vertical strips $V_\pm$ of width 1 conjugating F to the translation by 1. For further use we notice that, by symmetry, c_0 being real, we may assume that $\varphi_\pm(\overline{z}) = \overline{\varphi_\pm(z)}$. We also notice that these maps are unique up to post-composition by a real translation. These two maps are called respectively repelling (+) and attracting (-) Fatou coordinates. We normalize them as follows. For every $\kappa > 0$ and $R > 0$, consider two sectors $\Sigma_-(\kappa, R) = \{w : Re(w) > R - \kappa|Im(w)|\}$, $\Sigma_+(\kappa, R) = \{w : Re(w) < -R + \kappa|Im(w)|\}$. Then for any $\kappa > 0$ there is a big enough $R(\kappa)$, such

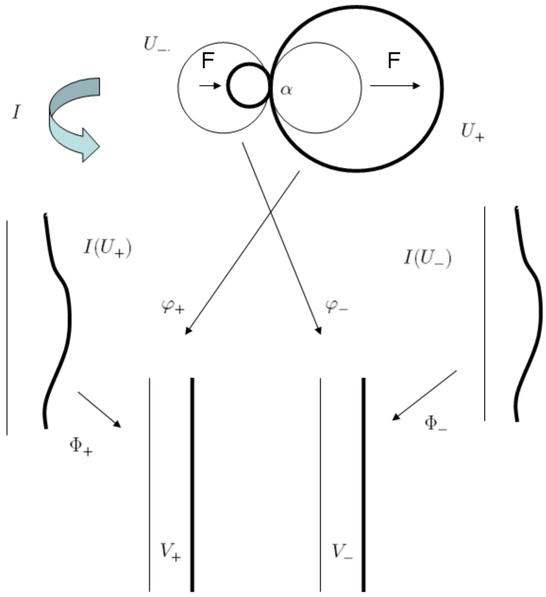

FIGURE 4 – Fatou coordinates

that, if we introduce two sectors $\Sigma_\pm(\kappa) = \Sigma_\pm(\kappa, R(\kappa))$, then $\varphi_\pm = \Phi_\pm \circ I$, where

$$\Phi_\pm(w) = w - A \log_\pm(w) + C_\pm + o(1) \tag{3}$$

as $w \to \infty$ within $\Sigma_\pm(\kappa)$ respectively. We specify the constants $C_\pm$ and the log-branches in such a way, that $\Phi_\pm$ are real for real w. Namely, we choose $C_- = 0$ and $\log_-$ to be the standard log-branch in the slit plane $\mathbf{C} \setminus \{\mathbf{x} \leq \mathbf{0}\}$. In turn, let $C_+ = iA\pi$ and $\log_+$ to be a branch of *log* in $\mathbf{C} \setminus \{\mathbf{x} \geq \mathbf{0}\}$ so that $\log_+(w) = \log|w| + i\pi$ for $w < 0$.

Now, we extend $\varphi_\pm$ in the following way. Since

$$\varphi_-(F(z)) = T_1(\varphi_-(z)) \tag{4}$$

where T_σ denotes the translation by σ, and since every orbit converging to α passes through U_- exactly once, φ_- extends uniquely to Δ to an holomorphic function still satisfying (4).

It is seen from (4), that the map $\varphi_- : \Delta \to \mathbf{C}$ is a branched covering, with the critical points at 0 and all its preimages in Δ by F^n, $n > 0$, and the critical values at the real numbers $\varphi_-(0) - n$, $n \geq 0$. In particular, there exists a simply-connected domain $\Omega_- \subset \Delta \cap \mathbf{H}^+$, where $\mathbf{H}^+$ is the upper half-plane, such that $\varphi_- : \Omega_- \to \mathbf{H}^+$ is a holomorphic homeomorphism. Moreover, the intersecrion of the boundary of Ω_- with $\mathbf{R}$ is the interval $(0, \alpha)$.

Concerning the repelling Fatou coordinate it is best to consider $\psi_+ = \varphi_-^{-1} : V_+ \to U_+$. The functionnal relation is now

$$\psi_+(T_1(z)) = F(\psi_+(z)) \tag{5}$$

and we can extend ψ_+ to an entire function by putting, for $n \in \mathbf{Z}$, $\psi_+(T_n(z)) = F^n(\psi_+(z))$ for $z \in V_+$. There exists a simply connected domain $\Omega_+ \subset \mathbf{H}^+$, such that $\psi_+ : \Omega_+ \to \mathbf{H}^+$ is a homeomorphism.
Let now σ be a real number. We define the Lavaurs map g_σ on the component Δ of the interior of the filled-in Julia set of f_{c_0} by

$$g_\sigma = \psi_+ \circ T_\sigma \circ \varphi_-.$$

The "raison d'être" of this definition is the following theorem due to Douady and Lavaurs ([2]) :

Theorem 2 *:There exists a sequence of positive ϵ_n converging to 0 and a sequence of positive integers N_n such that*

$$g_\sigma(z) = \lim_{n\to\infty} f^{kN_n}_{c_0+\epsilon_n}(z) \tag{6}$$

uniformly on compact sets of Δ.
Using (3) with the constants $C_\pm$ and the log-branches specified as above, it is easy to get, that, for every κ, if w tends to ∞ in $\Sigma(\kappa) := \Sigma_-(\kappa) \cap \Sigma_+(\kappa) \cap \mathbf{H}^+$, then $g_\infty(w) := I \circ g_\sigma \circ I^{-1}(w) = w + (\sigma - iA\pi) + O(\frac{1}{|w|})$, where A is real and positive. Therefore, for every real σ and every $\kappa > \kappa(\sigma)$, the inverse map g_∞^{-1} leaves the sector $\Sigma(\kappa)$ invariant and $g_\infty^{-n} \to \infty$ as $n \to \infty$. Coming back to the z-plane, we conclude that the branch $G = I^{-1} \circ g_\infty^{-1} \circ I$ of g_σ^{-1} leaves the set $S(\kappa) = I^{-1}(\Sigma(\kappa))$ invariant, and $G^n(z) \to \alpha$ as $n \to \infty$, for $z \in S(\kappa)$ and every $\kappa > \kappa(\sigma)$. We have, for $w \in \Sigma(\kappa)$:

$$G_\infty(w) := I \circ G \circ I^{-1}(w) = w + (-\sigma + iA\pi) + O(\frac{1}{|w|}), \tag{7}$$

Now, from the definition of $g_\sigma(z)$ and the global properties of the maps φ_- and ψ_+, it follows the existence of a simply-connected domain $\Omega_0 \subset \Omega_-$, which is mapped by g_σ homeomorphically onto $\mathbf{H}^+$ and such that $\alpha \in \bar{\Omega}_0$. Moreover, from the above description, $S(\kappa) \subset \Omega_0$, for every $\kappa > \kappa(\sigma)$. Therefore, the branch G of g^{-1} which is defined above, extends to a global univalent branch $G : \mathbf{H}^+ \to \Omega_0$ of g_σ^{-1}. Since $\Omega_0 \subset \mathbf{H}^+$, the iterates $G^n : \mathbf{H}^+ \to \Omega_0$, $n > 0$, converge uniformly on compact sets in $\mathbf{H}^+$ to a unique fixed point in $\bar{\Omega}_0$, which must be α.

Let us consider the continuous map $\sigma \mapsto g_\sigma(0) = \psi_+(\varphi_-(0) + \sigma)$: if σ runs in the interval $I = \{\varphi_+(x) - \varphi_-(0) : x \in U_+ \cap \mathbf{R}\}$, then $g_\sigma(0)$ runs over $U_+ \cap \mathbf{R}$. It is thus clear, and this is the key point in the proof, that we can choose σ in such a way that $g_\sigma(0)$ is a preimage of α : there is $j \geq 1$, such that $f^j_{c_0} \circ g_\sigma(0) = \alpha$. Since 0 is a critical point for g_σ, taking the inverse image by $f^j_{c_0} \circ g_\sigma$ has the same effect as multiplying the number of petals by d and we may state :

Lemma 2.1 *There exists an infinite iterated function system defined on a small compact neighborhood B_0 of zero and generated by some holomorphic branches of $f_{c_0}^{-1}$ and g_σ^{-1} such that its limit set has Hausdorff dimension bigger than $2d/(d+1)$.*

Comment 3 *In fact, the limit set is a subset of a so-called Julia-Lavaurs set denoted by $J_{c_0,\sigma}$. It is defined as follows. The map g_σ can be extended in a natural way from Δ to the interior of the filled-in Fatou set of f_{c_0} : if $f_{c_0}^k(z) \in \Delta$, set $g_\sigma(z) = g_\sigma \circ f_{c_0}^k(z)$. Then $J_{c_0,\sigma}$ is simply the closure of the set of points z for which there exists $m \in \mathbf{N}$ such that $g_\sigma^m(z)$ is defined and belongs to $J(f_0)$.*

This lemma together with the above discussion implies the theorem. Indeed, by a general property of iterated function systems [7], there exists its finite subsystem with the Hausdorff dimension of its limit set bigger than $2d/(d+1)$. On the other hand, the finite iterated function system persists for $f_{c_0+\epsilon}$ by (6). To be more precise, if $\{I_j : B_0 \to X_j, 1 \le j \le j_0\}$ is this finite iterated function system, then each I_j can be extended to a univelent map to a fixed neighborhood Y of B_0 as $I_j : Y \to Y_j$. Consider the inverse univalent map $I_j^{-1} : Y_j \to Y$. Since the convergence in (6) is uniform on compacts in Δ, for every ϵ_n small enough there is some integer $N_j > 0$ and a compact set $X_{j,n}$, so that $f_{c_0+\epsilon_n}^{N_j} : X_{j,n} \to B_0$ is univalent, too. Now it is clear, that, for every ϵ_n small enough, the non-escaping set K_n of the dynamical system which consists of a finitely many maps $f_{c_0+\epsilon_n}^{N_j} : X_{j,n} \to B_0$, $1 \le j \le j_0$, has the Hausdorff dimension which is bigger than $2d/(d+1)$. On the other hand, K_n must lie in the Julia set of $f_{c_0+\epsilon_n}$ because some iterate of the map $f_{c_0+\epsilon_n}^{N_1 N_2 \ldots N_{j_0}}$ leaves the set K_n invariant and is expanding on it.

Proof of Lemma 2.1. As the first step, let us fix a small enough closed ball B_0 around zero, so that it does not contain points of the postcritical set of f_{c_0}. There exists its preimage B' by F^{-1} in $\Delta \cap \mathbf{H}^+$. Then we can apply to B' the maps G^n, $n > 0$. By the above, $B'_n = G^n(B')$ are pairwise disjoint, compactly contained in Δ, and $B'_n \to \alpha$ as $n \to \infty$. Now, for every $n \ge n_0$, so that B'_n lies in a small enough neighborhood U of α, we make "clones" of B'_n in $U \cap \Delta$ applying to it F^r, $r \in \mathbf{Z}$, where F^r for $r < 0$ is a well-defined in $U \cap \Delta$ branch which fixes α . We obtain the sets $B'_{n,r} = F^r(B'_n)$. On the second step, we consider the map $f_{c_0}^j \circ g_\sigma$ from a neighborhood V of 0 onto U, This map is a remified cover with the only remification point at 0 of order d. Let $U^* = U \setminus \{x \ge \alpha\}$, and $V^* = V \cap \{z : Arg(z) \in (0, 2\pi/d)\}$. Denote by h a branch of $(f_{c_0}^j \circ g_\sigma)^{-1}$ from U^* onto V^*. Let $B_{n,r} = h(B'_{n,r})$. We obtain a system of holomorphic maps $\mathbf{\Psi} = \{\psi_{n,r} : B_0 \to B_{n,r}\}$, where $\psi_{n,r} = h \circ F^r \circ G^n \circ F^{-1}$, $r \in \mathbf{Z}, n \ge n_0$. If the neighborhood U is chosen small enough, the maps $\psi_{n,r}$ extend to univalent maps in a fixed neighborhood $\tilde{B}$ of B_0 into itself. In particular, the compact sets $B_{n,r}$ are pairwise disjoint and compactly contained in B_0. Now, it is quite standard to check that $\mathbf{\Psi}$ form a conformal infinite iterated function system in the sense of [7] (strictly speaking, in the hyperbolic metric of $\tilde{B}$, which is equivalent to the Eucledian one on B_0 though). Let us calculate the parameter $\theta = \inf\{t : \psi(t) < \infty\}$ of $\mathbf{\Psi}$, where $\psi(t) = \sum_{(n,r)} \max_{z \in B_0} |\psi'_{n,r}(z)|^t$. The map $\psi_{n,r} = h \circ I^{-1} \circ F_\infty^r \circ G_\infty^n \circ I \circ F^{-1}$. Here F^{-1} is a univalent map of a neighborhood $\tilde{B}$ of B_0 into Δ. Now, routine and well-know calculations based on (2)-(7) show (see e.g. [10]), that, for all $r \in \mathbf{Z}, n \geq n_0$ and some C, which depends only on a compact set in $\mathbf{H}^+ \cap \Delta$, from which w is taken, $C^{-1}|r + (-\sigma + i\pi A)n| \leq |F_\infty^r \circ G_\infty^n(w)| \leq C|r + (-\sigma + i\pi A)n|$, and $C^{-1} \leq |(F_\infty^r \circ G_\infty^n)'(w)| \leq C$. On the other hand, the map h is a composition of a univalent map with an inverse branch of $z^{1/d}$. This gives us : $C_1^{-1}|r + (-\sigma + i\pi A)n|^{-1-1/d} \leq |\psi'_{n,r}(z)| \leq C_1|r + (-\sigma + i\pi A)n|^{-1-1/d}$, for some C_1 and every $z \in B_0$. It follows that the series for $\psi(t)$ converges if and only if $t > \theta = 2d/(d+1)$, and $\psi(\theta) = \infty$. Hence [7], the Hausdorff dimension of the limit set of $\mathbf{\Psi}$ is strictly bigger than $2d/(d+1)$.

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Références

[1] A.Avila and M.Lyubich. Examples of Feigenbaum Julia sets with small Hausdorff dimension. *Dynamics on the Riemann Sphere, in honor of B.Branner*, 1 :71–87, 2006.

[2] A.Douady. Does a Julia set depend continuously on the Polynomial ? *Proceedings of Symposia in Applied Mathematics*, 49 :91–135, 1994.

[3] A.Douady and J.Hubbard. Etude dynamique des Polynômes complexes : Première partie. *Publications mathématiques d'Orsay*, 84-02 :1–75, 1984.

[4] G.Levin and G.Swiatek. Hausdorff dimension of Julia sets of Feigenbaum type. *Communications in Mathematical Physics*, 258 :135–148, 2005.

[5] G.Levin and G.Swiatek. Measure of the Julia set of the Feigenbaum map with high criticality. *Ergodic Theory and Dynamical Systems*, 30 :855–875, 2009.

[6] J.Graczyk and G.Swiatek. Generic Hyperbolicity in the Logistic Family. *Annals of Mathematics*, 146 :1–52, 1997.

[7] R.D. Mauldin and M. Urbanski. Dimension and measure in infinite iterated function systems. *Proc. London Math. Soc.*, 73 :105–154, 1996.

[8] M.Lyubich. Dynamics of quadratic polynomials. *Acta Mathematica*, 178 :185–297, 2007.

[9] M.Shishikura. The Hausdorff Dimension of the Boundary of the Mandelbrot set and Julia Sets. *Annals of Mathematics*, 147 :225–267, 1998.

[10] M.Zinsmeister. Fleur de Leau-Fatou et dimension de Hausdorff. *Comptes-rendus de l'Académie des Sciences*, 326 :1227–1232, 1998.

[11] O.Kozlovsky, W.Shen, and S. van Strien. Density of Hyperbolicity in Dimension one. *Annals of mathematics*, 166 :145–182, 2007.

[12] X.Buff and A.Chéritat. Ensembles de Julia quadratiques de mesure strictement positive. *Comptes-rendus de l'Académie des Sciences*, 341 :669–674, 2005.

[13] X.Buff, R.Devaney, A.Douady, and P.Sentenac. Baby Mandelbrot sets are born in cauliflowers. *London Mathematical Society Lecture Notes Series*, 274 :19–36, 2000.

[14] Anna Zdunik. Parabolic orbifolds and the dimension of the maximal measure for rational maps. *Inventiones Mathematicae*, 99 :627–649, 1990.